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$W_{1+\infty}$ as a Discretization of Virasoro Algebra

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Abstract

It is shown that the $W_{1+\infty}$ algebra is nothing but the simplest subalgebra of a q-discretized Virasoro algebra (D–Virasoro), in the language of the KP hierarchy explicitly.

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1 Introduction

In the research of integrable systems, discretization of variables provides important information about the systems. Stability under discretization characterizes a certain class of integrable systems which is completely integrable. For example every soliton equation of the KP hierarchy [1, 2] has a discrete analog which is common to all equations. Namely a single bilinear difference equation of Hirota reproduces every soliton equation of the KP hierarchy by taking certain continuous limit of variables [3]. This remarkable property of soliton type equations should be compared with a generic case in which discretization of variables in general make nonlinear equations create chaos [4]. This means that there exist some symmetries which preserve integrability under the discretization of variables.

Discretization of differential operators also plays a key role in the q-deformed conformal field theories. The q-deformed Knizhnik-Zamolodchikov (q-KZ) equation by Frenkel and Reshetikhin [5] has been formulated by the consideration of representation theory of $U_q(\widehat{sl_2})$, and is essentially realized through discretization of variables of the original KZ equation.

We are interested in relations which remain true irrespective to continuous or discrete. Such relations will not only characterize completely integrable systems but also cast a light to understand the boundary between deterministic and nondeterministic nature of nonlinear equations.

In order to claim this idea we proposed, in a series of papers [6], a deformation of the Virasoro algebra which was realized by a q-discretization of differential operators and called it D–Virasoro. This algebra was shown to admit free field representation both of fermionic and bosonic [7]. It was constructed such that the Virasoro algebra was reproduced in the continuous limit ($q \to 1$). Although it was derived as general as possible, there remains a problem to clarify the relation to other known symmetries.

In this paper we show explicitly that the simplest subalgebra of D–Virasoro is nothing but the $W_{1+\infty}$ algebra [8]. This result provides a new interpretation of the $W_{1+\infty}$ algebra. Namely it enables us to understand that the $W_{1+\infty}$ algebra emerges as a result of a proper discretization of the Virasoro algebra.

This paper is organized as follows: In Section 2, we review on the discretization of the Virasoro algebra. In Section 3, we apply it to the KP hierarchy. For the purpose of it, we first review on additional symmetry of the KP hierarchy which is found in [9] and generalized in [10]. Next we investigate how the flow on the universal Grassmann manifold induced by q-shift operators can be expressed in terms of the symmetry, and denote it by E_{mn}. Based on the results of [11], we show that the action of E_{mn} on the τ function can be realized by use of the q-shift operators and the vertex operators of the KP hierarchy. In the last part, we construct

operators by a proper combination of E_{mn}. They will be shown equivalent to generators of the simplest D–Virasoro subalgebra as the action on the τ function. Finally, by using the fact that the vertex operators give generators of the $W_{1+\infty}$ algebra [12, 13], we find that the subalgebra is the $W_{1+\infty}$ algebra.

2 A brief review on D–Virasoro

In [6], we proposed the following algebra (without central extension);

$$\begin{aligned}\left[\mathcal{L}_m^{(n,r;\pm)}, \mathcal{L}_{m'}^{(n',r;\pm)}\right] &= C\binom{m \;\; n+r}{m' \;\; n'+r}_{\pm} \mathcal{L}_{m+m'}^{(n+n'+r,r;\pm)} + C\binom{m \;\; n-r}{m' \;\; n'-r}_{\pm} \mathcal{L}_{m+m'}^{(n+n'-r,r;\pm)} \\ &\quad + C\binom{m \;\; n+r}{m' \;\; -n'+r}_{\pm} \mathcal{L}_{m+m'}^{(n-n'+r,r;\pm)} + C\binom{m \;\; n-r}{m' \;\; -n'-r}_{\pm} \mathcal{L}_{m+m'}^{(n-n'-r,r;\pm)} \qquad (1)\end{aligned}$$

for each value of r. The double signs on both sides correspond each other. The structure constant C is given by

$$C\binom{m \;\; n+r}{m' \;\; n'+r}_{\pm} \equiv \frac{\left[\frac{(n+r)m'-(n'+r)m}{2}\right]_{-} [n+n'+r]_{\mp}}{2\, [n]_{\mp}\, [n']_{\mp}\, [r]_{\pm}} \;, \qquad (2)$$

where

$$[x]_{+} \equiv \frac{q^x + q^{-x}}{2} \;,\; [x]_{-} \equiv \frac{q^x - q^{-x}}{q - q^{-1}} \;.$$

The generators $\mathcal{L}_m^{(n,r;\pm)}$ are realized as

$$\mathcal{L}_m^{(n,r;\pm)} = z^m \frac{\left[n\left(z\partial_z + \frac{m}{2}\right)\right]_{\mp}}{[n]_{\mp}} \frac{\left[rz\partial_z\right]_{\pm}}{[r]_{\pm}} \equiv z^m \left[z\partial_z + \frac{m}{2}\right]_{n;\mp} \left[z\partial_z\right]_{r;\pm} \;. \qquad (3)$$

In (3), $\mathcal{L}_m^{(n,r;\pm)}$ essentially consist of q-difference operators. Moreover, in the limit of $q \to 1$, $\mathcal{L}_m^{(n,r;\pm)}$ reduce to the Virasoro generators and (1) becomes the commutation relation of the Virasoro algebra. Then we named it D–Virasoro algebra (D-Vir).

We also have the central extended version of this algebra which are represented by free fields as follows;

$$\hat{\mathcal{L}}_m^{(n,r;\pm)} = \frac{1}{2} \sum_k A_{k,m-k}^{(n;\pm)}(q) \;\; : \alpha_{m-k}^{(r;\pm)} \; \alpha_k^{(r;\pm)} : . \qquad (4)$$

where $\alpha_k^{(r;\pm)}$ satisfy the following relations;

$$[\alpha_k^{(r;\pm)}, \alpha_{k'}^{(r;\pm)}]_{\pm} \equiv \alpha_k^{(r;\pm)} \alpha_{k'}^{(r;\pm)} \pm \alpha_{k'}^{(r;\pm)} \alpha_k^{(r;\pm)} = D_k^{(r;\pm)}(q)\; \delta_{k+k',0} \;, \qquad (5)$$

i.e. $\alpha_k^{(r;+)}$ denotes 'fermion', and $\alpha_k^{(r;-)}$ does 'boson', respectively. We get simple solutions for A and D;

$$A_{k,m-k}^{(n;\pm)} = -\left[\frac{2k-m}{2}\right]_{n;\mp}, \; D_k^{(r;\pm)} = [k]_{r;\pm} \;. \tag{6}$$

In the following discussion, we will not use the full structure of this algebra. As our attention is paid to its relation to the KP hierarchy, we only use a subalgebra D–Vir$^{(+,r=0)}$ of which generators $\{\mathcal{L}_m^{(n,0;+)}\}$ $(n \in \mathbf{Z}_{>0})$ are realized by use of the ordinary fermion like as the KP hierarchy.

3 Action of D–Virasoro to the KP hierarchy

3.1 Additional symmetries of the KP hierarchy

Let us start with the KP hierarchy with Lax representation [1, 2];

$$\begin{aligned} &\partial_l L = [L_+^l, L] \\ &L = W\partial W^{-1}, \qquad W = 1 + w_1(x)\partial^{-1} + w_2(x)\partial^{-2} + \cdots, \end{aligned} \tag{7}$$

where L_+ denotes the differential operator part of the pseudo-differential operator L, then $L = L_+ + L_-$, and $\partial = \partial/\partial x$, $\partial_l = \partial/\partial t_l$ $(l \in \mathbf{Z}_{>0},\ x = t_1)$. Any equation of (7) generates symmetry for other equations, and they are commutative in the sense that $[\partial_l, \partial_{l'}] = 0$. By removing the dressing, (7) becomes $[\partial_l - \partial^l, \partial] = 0$.

In addition to this symmetry it is known that other symmetries exist [9, 10]. First we introduce an operator

$$\Gamma = \sum_{r=1}^{\infty} r t_r \partial^{r-1}. \tag{8}$$

It is clear that $[\partial_l - \partial^l, \Gamma] = 0$. Then if we define $M = W\Gamma W^{-1}$, we can easily check $[\partial_l - L_+^l, M] = 0$. (The essential difference between M and L is in the expansion coefficients. The coefficients of M may depend on the KP time $\{t_l\}$ explicitly.) By combining this equation with (7) and generalizing them, we get

$$[\partial_l - L_+^l, M^k L^m] = 0 \qquad \forall\ k \in \mathbf{Z}_{>0}, m \in \mathbf{Z} \tag{9}$$

If we introduce new variables t_{mk} satisfying the following equation as

$$\partial_{mk} L = -[(M^k L^m)_-, L] \qquad \partial_{mk} = \frac{\partial}{\partial t_{mk}}, \tag{10}$$

or in terms of W operators;

$$\partial_{mk}W = -(M^k L^m)_- W, \tag{11}$$

we can prove that $\{\partial_{mk}\}$ commutes with the KP flow, i.e. $[\partial_{mk}, \partial_l] = 0$. In this sense the flow $\{\partial_{mk}\}$ were called *additional* symmetries of the KP hierarchy[†] [12]. But the new flow themselves do not commute with each other: Since the operators L and M are canonically conjugate, i.e. $[L, M] = 1$, we can consider the homomorphism such as $L \mapsto \partial$ and $M \mapsto x$. This mapping enables us to calculate the commutation relation of $\{\partial_{mk}\}$. Actually, we get

$$[\partial_{mk}, \partial_{m'k'}] = \sum_{j=1}^{\infty}\left\{\begin{pmatrix} m \\ j \end{pmatrix}\begin{pmatrix} k \\ j \end{pmatrix} - \begin{pmatrix} m' \\ j \end{pmatrix}\begin{pmatrix} k' \\ j \end{pmatrix}\right\} j!\ \partial_{m+m'-j,k+k'-j}. \tag{12}$$

In a simple case as $\{\partial_{m+1,1}\}$, it is well-known that they form an algebra isomorphic to Virasoro algebra (without central extension). Remark that (10) and (11) have the degree of freedom for the gauge choice. But we will fix the gauge in the above form in the following discussion.

Now we show the action of the new symmetry to an element of the universal Grassmann manifold (UGM). Let H be the space of formal expansions $H = \{f(z) = \sum f_k z^k\}$. If a subspace $V \subset H$ has a natural bijection $V \to H_+$, where H_+ is the positive power part of H, then UGM is defined by UGM $= \{V \subset H | V \simeq H_+\}$. It is known that there is a *monic* z-operator[‡] $G = 1 + \sum a_{ij} z^i \partial_z^j$ such that $V = GH_+$. So generically, for $v \in V$ and $h_+ \in H_+$, there is a monic operator $W(t^*, \partial_z, z)$ such as $e^{-\xi^*}v = W(t^*, \partial_z, z)h_+$ where $\xi^* = \xi(t^*, z) = \sum_{r=2}^{\infty} t_r z^r$. If we consider the replacement as $z \mapsto \partial$, $\partial_z \mapsto x$ and write the factors of the operators in an inverse order, we see that z-operators and pseudo-differential operators are related as an anti-isomorphism; $G(\partial_z, z) \mapsto G(x, \partial) = \sum a_{ij} x^j \partial^i$. By using this formula, we can prove that $W(t^*, x, \partial)$ is nothing the dressing operator of the KP hierarchy. (see [14] for more detail)

We can consider that the additional symmetry flow on UGM is given by the operators $\partial_{mk} : V \to H/V$. Namely

$$\begin{aligned}
\partial_{mk}\ e^{-\xi^*}v \ &= \ \partial_{mk}\ W(\partial_z, z)h_+ \\
&= -W\Big(L(\partial_z, z)^m M(\partial_z, z)^k\Big)h_+ \qquad (\mathrm{mod}\ V) \\
&= -z^m\left(\partial_z + \sum_{2}^{\infty} r t_r z^{r-1}\right)^k e^{-\xi^*}v = -e^{-\xi^*}z^m\partial_z^k\ v.
\end{aligned}$$

[†]If $k = 0$, ∂_{m0} is ∂_m for $m > 0$.

[‡]In a standard form, z-operators are defined by

$$G = G(\partial_z, z) = \sum_{i\le 0, j\ge 0} a_{ij} z^i \partial_z^j.$$

The last equality can be obtained by induction. If we use $[\partial_{mk}, \partial_i] = 0$, the l.h.s. becomes $\partial_{mk}\ e^{-\xi^*}v = e^{-\xi^*}\partial_{mk}\ v$. Then dividing by $e^{-\xi^*}$ on both sides, we get

$$\partial_{mk}\ v = -z^m \partial_z^k\ v. \tag{13}$$

In general, the algebra which consists of the operator $\{z^m \partial_z^k\}$ is called the W-infinity algebra. Then the flow ∂_{mk} is nothing but the action of the W-infinity on UGM.

3.2 Flow induced by shift operators

To apply the result of above discussion to our theory, we first expand q-shift operators $q^{nz\partial_z}$ with respect to the differential operators,

$$q^{nz\partial_z} = \sum_{j=0}^{\infty} \frac{(n\lambda)^j}{j!}(z\partial_z)^j = \sum_{j=0}^{\infty} \frac{(n\lambda)^j}{j!} \sum_{k=0}^{\infty} c_{jk}\ z^k \partial_z^k\ , \tag{14}$$

where $\lambda = \ln q$ $(q^N \neq 1;\ \forall N \in \mathbf{Z}_{>0})$ and the coefficients c_{jk} is given by

$$c_{jk} = \sum_{\alpha=1}^{k} \frac{(-1)^{k-\alpha}\ \alpha^j}{(k-\alpha)!\ \alpha!} \qquad (j,k \geq 1), \qquad c_{l0} = c_{0l} = \delta_{l,0}\ . \tag{15}$$

Remark that we can solve (14) conversely,

$$z^k \partial_z^k = \sum_{j=0}^{\infty} d_{kj}\ (z\partial_z)^j = \sum_{j=0}^{\infty} d_{kj}\ j!\ n^{-j} \oint \frac{d\lambda}{2\pi i}\ \lambda^{-j-1}\ e^{n\lambda z \partial_z}, \tag{16}$$

where d_{kj} is the inverse of the $(\infty \times \infty)$ matrix c_{jk} and satisfy the following relations;

$$d_{kj} = (-1)^{k+j}\ \{d_{k-1,j-1} + (k-1)\ d_{k-1,j}\} \qquad (j,k \geq 1), \qquad d_{l0} = d_{0l} = \delta_{l,0}. \tag{17}$$

Now we consider a form as

$$E_{mn} \equiv -q^{\frac{mn}{2}} \sum_{j=0}^{\infty} \frac{(n\lambda)^j}{j!} \sum_{k=0}^{\infty} c_{jk}\ \partial_{m+k,k}\ . \tag{18}$$

From (13) and (14), we understand that the flow E_{mn} on UGM is equivalent to the action of q-shift operator, that is

$$E_{mn}\ v = z^m q^{n(z\partial_z + \frac{m}{2})}\ v. \tag{19}$$

So if we remember (3), we can write the action of D-Vir$^{(+,0)}$ on UGM in terms of additional symmetry flow;

$$\mathcal{L}_m^{(n,0;+)}\ v = \left[\frac{E_{mn} - E_{m,-n}}{q^n - q^{-n}} \right]\ v. \tag{20}$$

In this form, we see that $\mathcal{L}_m^{(n,0;+)}$ is nothing but the q-difference operator of the spectral parameter z of the KP hierarchy. It is easy to check that in the $q \to 1$ limit,

$$\mathcal{L}_m^{(n,0;+)}\ v = z^m \left[z\partial_z + \frac{m}{2} \right]_{n;-} v \ \longrightarrow\ -\partial_{m+1,1}\ v.$$

Next we consider the action of E_{mn} on τ function. Based on the above discussion, we first write down the action of the operator $z^m\partial_z^k$ on the wave function $w(z) = We^{\xi(t,z)}$ in terms of pseudo-differential operators;

$$z^m\partial_z^k w(z) = W(x,\partial)\Gamma^k\partial^m e^{\xi}. \tag{21}$$

After multiplying the adjoint wave function $w^*(z) = (W^*)^{-1}e^{-\xi(t,z)}$ from the right on both sides, we integrate along a contour around $z = \infty$;

$$\oint \frac{dz}{2\pi i} \left(z^m\partial_z^k w(z) \right)\ w^*(z) = \oint \frac{dz}{2\pi i} \left(W(x,\partial)\Gamma^k\partial^m e^{xz} \right) \left((W^*)^{-1}e^{-xz} \right), \tag{22}$$

and look at how each side is expressed by using the τ function.

○ *The l.h.s. of (22)*

The wave functions can be written as

$$w(z,t) = \frac{V(z)\ \tau}{\tau}, \qquad w^*(z,t) = \frac{V^*(z)\ \tau}{\tau}, \tag{23}$$

where the vertex operators V and V^* are defined as

$$V(z) = \exp\left(\sum_{r=1}^{\infty} z^r t_r \right)\ \exp\left(-\sum_{r=1}^{\infty} \frac{1}{r} z^{-r}\partial_r \right), \tag{24a}$$

$$V^*(z) = \exp\left(-\sum_{r=1}^{\infty} z^r t_r \right)\ \exp\left(\sum_{r=1}^{\infty} \frac{1}{r} z^{-r}\partial_r \right), \tag{24b}$$

respectively. They satisfy the following anti-commutation relation as

$$\{V(z), V^*(\zeta)\} = \delta\left(\frac{z}{\zeta}\right). \tag{25}$$

The δ function is defined by formal expansions as

$$\delta(z) = \sum_{k\in\mathbf{Z}} z^k. \tag{26}$$

If we use (25) and the bilinear identity for the wave functions;

$$\oint \frac{dz}{2\pi i}\ w(z,t)\ w^*(z,t') = 0, \tag{27}$$

we can write the bilinear form $w(z)w^*(\zeta)$ by means of the vertex operators and the τ function as

$$V(z)V^*(\zeta)\ (-\partial \ln \tau).$$

Hence the l.h.s. of (22) is written as

$$\oint \frac{dz}{2\pi i}\ \left(z^m \partial_z^k V(z)\right)\ V^*(z)\ (-\partial \ln \tau). \tag{28}$$

○ *The r.h.s. of (22)*

We use the following lemma [11, 12];

Lemma *For any two pseudo-differential operators P and Q,*

$$\oint \frac{dz}{2\pi i}\ (Pe^{zx})(Qe^{-zx}) = \mathrm{res}_\partial\ PQ^*, \tag{29}$$

where $\mathrm{res}_\partial \sum a_k \partial^k = a_{-1}$. □

Then the r.h.s. of (22) becomes

$$\begin{aligned} \mathrm{res}_\partial\ W\Gamma^k \partial^m W^{-1} &= \mathrm{res}_\partial\ M^k L^m = \mathrm{res}_\partial\ (M^k L^m)_- W \\ &= -\mathrm{res}_\partial\ \partial_{mk} w(z) = -\partial_{mk} w_1 = \partial_{mk}(\partial \ln \tau). \end{aligned} \tag{30}$$

From (28) and (30), we find that the flow ∂_{mk} induced by the action of $z^m \partial_z^k$ on UGM acts on the τ function in the following way [11];

$$\partial_{mk}\ \tau = -\left[\oint \frac{dz}{2\pi i}\ z^m \frac{\partial^k V(z)}{\partial z^k} V^*(z)\right]\ \tau. \tag{31}$$

This expression enables us to write the flow E_{mn} on the τ function in more familiar form. Actually, by using (19) and (31) we obtain

$$\begin{aligned} E_{mn}\ \tau &= \left[\oint \frac{dz}{2\pi i}\ z^m \left(q^{n(z\partial_z + \frac{m}{2})}\ V(z)\right)\ V^*(z)\right]\ \tau \\ &= \left[\frac{q^{\frac{nm}{2}}}{1-q^{-n}} \oint \frac{dz}{2\pi i}\ z^m X(q^n z, z)\right]\ \tau. \end{aligned} \tag{32}$$

where the vertex operator $X(z,\zeta) \equiv : V(z)V^*(\zeta) :$.

3.3 $W_{1+\infty}$ as D-$\mathbf{Vir}^{(+,0)}$

The final result of the preceding part is very important for our discussions: In the theories of the KP hierarchy, the vertex operators relate to the generators of the $W_{1+\infty}$ algebra. Then if we can show that the transformation $\tau \to \tau + E_{mn}\tau$ relates to D–Virasoro, we understand the relation between $W_{1+\infty}$ and D–Virasoro. So we first investigate how the flow E_{mn} connects with D–Virasoro.

We consider the Fock representation of the KP hierarchy [2]. Let ψ_l and ψ_l^* ($l \in \mathbf{Z} + \frac{1}{2}$) be free fermions such as $\{\psi_l, \psi_{l'}^*\} = \delta_{l+l',0}$ and $\{\psi_l, \psi_{l'}\} = \{\psi_l^*, \psi_{l'}^*\} = 0$. The vacuum $\langle 0|$ and $|0\rangle$ are defined by

$$\psi_l|0\rangle = 0 \ (l < 0), \qquad \psi_l^*|0\rangle = 0 \ (l > 0), \tag{33a}$$

$$\langle 0|\psi_l = 0 \ (l > 0), \qquad \langle 0|\psi_l^* = 0 \ (l < 0). \tag{33b}$$

In this representation, the τ function can be written as

$$\tau(t, g) = \langle 0|e^{H(t)} g|0\rangle \qquad g \in GL(\infty) \tag{34}$$

where

$$H(t) = \sum_{r=1}^{\infty} H_r t_r, \quad H_r = \sum_{l \in \mathbf{Z}+\frac{1}{2}} :\psi_l \psi_{r-l}^*:,$$

or equivalently

$$:\psi(z)\psi^*(z):= \sum_{r \in \mathbf{Z}} H_r z^{-r-1},$$

$$\psi(z) = \sum_{l \in \mathbf{Z}+\frac{1}{2}} \psi_l z^{-l-\frac{1}{2}}, \qquad \psi^*(z) = \sum_{l \in \mathbf{Z}+\frac{1}{2}} \psi_l^* z^{-l-\frac{1}{2}}.$$

The action of $:\psi(z)\psi^*(\zeta):$ on the (neutral) state $g|0\rangle$ corresponds to the action of the operator

$$Y(z, \zeta) \equiv \frac{1}{z - \zeta}(X(z, \zeta) - 1) \tag{35}$$

on $\tau(t, g)$, that is

$$Y(z, \zeta)\ \tau(t, g) = \langle 0|e^{H(t)} :\psi(z)\psi^*(\zeta): g|0\rangle. \tag{36}$$

If we set a form as

$$L_m^{(n)} = \frac{1}{q^n - q^{-n}} \left\{ q^{-n} E_{m-1,n} - q^n E_{m-1,-n} - \frac{\cosh(\lambda n(m-1)/2)}{\sinh(\lambda n/2)}\, \delta_{m,1} \right\}, \tag{37}$$

the action of $L_m^{(n)}$ to the τ function becomes

$$\begin{aligned}
L_m^{(n)}\ \tau &= \oint \frac{dz}{2\pi i}\ \langle 0|e^{H(t)} : \Big(\mathcal{L}_m^{(n,0;+)}\psi(z)\Big)\ \psi^*(z) : g|0\rangle \\
&= \langle 0|e^{H(t)}\Big(-\frac{1}{2}\sum_l \Big[\frac{2l-m+1}{2}\Big]_{n;-} : \psi_l\psi^*_{m-l} :\Big)g|0\rangle \\
&= \langle 0|e^{H(t)}\hat{\mathcal{L}}_m^{(n,0;+)}g|0\rangle, \qquad (38)
\end{aligned}$$

where $\hat{\mathcal{L}}_m^{(n,0;+)}$ is given by (4). We can easily check the following identity;

$$\Big[L_m^{(n)}, L_{m'}^{(n')}\Big]\ \tau = \langle 0|e^{H(t)}\Big[\hat{\mathcal{L}}_m^{(n,0;+)}, \hat{\mathcal{L}}_{m'}^{(n',0;+)}\Big]\ g|0\rangle. \qquad (39)$$

This means that $\{L_m^{(n)}\}$ is nothing but the generators of D-Vir$^{(+,0)}$ ($\subset D$-Vir) as the action on $\tau(t,g)$.

Since the generator of D-Vir$^{(+,0)}$ is represented by using the vertex operators, we can investigate the relation between D-Vir$^{(+,0)}$ and $W_{1+\infty}$. Let us set $\{W_p^{(k)};\ p\in\mathbf{Z}, k\in\mathbf{Z}_{>0}\}$ as generators of $W_{1+\infty}$. It is known that the generators of $W_{1+\infty}$ can also be written by using the vertex operators [12, 13];

$$W^{(k)}(z) = \sum_{p=-\infty}^{\infty} z^{-p-k}\ W_p^{(k)} = \Big(\frac{\partial}{\partial\zeta}\Big)^{k-1} Y(z,\zeta)\ \Big|_{\zeta=z}. \qquad (40)$$

Combining this equation with (32) and (37), we verify the relation between $L_m^{(n)}$ and $W_m^{(k)}$ as

$$\begin{aligned}
L_m^{(n)} &= -\sum_{k=1}^{\infty}\frac{e^{\frac{i\pi}{2}k}}{(k-1)!}\ \frac{\Big(2\sinh(\lambda n/2)\Big)^{k-1}\cosh\Big(\{\lambda n(m+k+1)-i\pi k\}/2\Big)}{\sinh\lambda n}\ W_m^{(k)} \\
&\equiv \sum_{k=1}^{\infty}(s_m)_{nk}\ W_m^{(k)} \qquad (41)
\end{aligned}$$

for fixed m. It is also written in the matrix form as

$$\mathbf{L}_m = \mathbf{S}_m\mathbf{W}_m, \qquad (42)$$

where the matrices are defined by

$$\mathbf{L}_m = {}^T(L_m^{(1)}, L_m^{(2)}, \cdots),\quad \mathbf{W}_m = {}^T(W_m^{(1)}, W_m^{(2)}, \cdots),\quad \mathbf{S}_m = (s_m)_{nk} \quad (n,k\in\mathbf{Z}_{>0}),$$

respectively. Since we can consider the following correspondence as

$$z^{m+k}\partial_z^k\ v \longleftrightarrow W_m^{(k)}\ \tau\ \ ;\ \ z^m\Big[z\partial_z + \frac{m}{2}\Big]_{n;-}\ v \longleftrightarrow L_m^{(n)}\ \tau, \qquad (43)$$

we can say $\mathbf{S}_m$ has the inverse similar to c_{jk} in (15) (see (16)). Then (42) is also written the form as $\mathbf{W}_m = \mathbf{S}_m^{-1}\mathbf{L}_m$. Hence we conclude that $\{L_m^{(k)}\}$ is another realization of $\{W_m^{(k)}\}$, that is D-Vir$^{(+,0)}$ is nothing but $W_{1+\infty}$. Though we sum up the index k in (41), i.e. we forget the information on the conformal spins, we recover their dependence via the index n of $\{L_m^{(n)}\}$. Mathematically, n decides the power of q, in other words, the interval of discretization. But from the field theoretical point of view, we can say the difference between D-Vir$^{(+,0)}$ and $W_{1+\infty}$ owes to the difference of the corresponding currents as follows:

$$D\text{-Vir}^{(+,0)} \quad \rightarrow \quad L^{(n)}(z) = \sum_m z^{-m-1}\ L_m^{(n)} \tag{44a}$$

$$W_{1+\infty} \quad \rightarrow \quad W^{(k)}(z) = \sum_m z^{-m-k}\ W_m^{(k)}. \tag{44b}$$

4 Concluding remarks

In this paper, we have found that the simplest D–Virasoro subalgebra, D-Vir$^{(+,0)}$, can be regarded as the $W_{1+\infty}$ algebra. The implication of this fact is very impressive: The W-infinity symmetry is considered as a universal symmetry structure for integrable systems in a sense that such symmetry often appears in various theories. On the other hand, as mentioned at the beginning, some discretization of the variables preserve integrability of the system. Therefore D-Vir$^{(+,0)}$ is not only a deformation of the Virasoro algebra but a discretization of the spectral parameter which preserves integrability of the KP hierarchy. This point of view must provide useful tools for understanding structure of integrable systems, such as the soliton theories, 2D gravity, etc.

The following is devoted to some remarks on the subject:

(i) As mentioned above, we have not used the whole structure of D-Vir. The remaining part must also have rich structure:

∘ If we consider D-Vir$^{(+,r\neq 0)}$ algebra, the fermion is no longer unique constituent. In this case it is natural to think that the representation is also "discretized", i.e. we must consider deformation of the KP hierarchy itself. It is interesting whether such system is still integrable or not.

∘ Another type of D–Virasoro subalgebra, D-Vir$^{(-)}$, is also related with the W-infinity algebra. From the fact that the generator of D-Vir$^{(-,r=1)}$ is realized by ordinary free bosons in (4), it is reasonable to guess that it is related to the W_∞ algebra [15].

(ii) Though we made no mention of in this paper, there exists structure of the Moyal bracket

[16] behind D–Virasoro [6]. The structure is so huge that it contains not only D–Virasoro but also deformation of some infinite dimensional Lie algebras [15, 17]. It suggests that the Moyal structure must have many informations on integrability. Therefore study of the symmetry associated with this structure seems valuable to have general view of integrable systems.

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